

Effective Subspace for Pure State Transfer Induced by Measurements

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One main goal of quantum control is to steer a quantum system toward an expected state or dynamics. For measurement-induced quantum control, measurements serve as the only control, which is like the cases in quantum Zeno and anti-zeno effects. In this paper, this scenario will be investigated in a general N -level quantum system. It is proved that, when the initial and expected states are both pure, the control space could be reduced to an effective subspace spanned by these two states only. This result will greatly simplify the measurement-induced control strategy of an N -level quantum system.

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I. INTRODUCTION

Quantum control has drawn much attention both theoretically and experimentally[1–5] due to its potential applications. The goal of quantum control is usually to steer a quantum system into an expected quantum state via external fields[6]. However, the control means are not limited to introducing control fields. With rapid development of the research field, various approaches were proposed to play the role of control[5]. The quantum measurement, as one of the necessary criteria for a quantum computer, does not only serve as the read-out but also could be employed as the control[7].

The idea of utilizing measurements to drive quantum systems originates from the famous quantum Zeno effect which claims that repeated frequent measurement of an unstable state will keep it unchanged[8, 9]. The counterpart of Zeno effect, namely anti-Zeno effect, was also discussed[10, 11], and both of them were observed in an unstable system[12]. In contrast to control field schemes, measurement-based quantum control will introduce irreversible decoherence inevitably. By combining with coherent control, it can be applied to closed-loop molecular states control[13], quantum dynamics engineering[14], dephasing decoherence control[15] and remote state preparation[16]. Controllability analysis of quantum systems shows that measurements could transform a non completely controllable system into a controllable one while unitary evolution only could not achieve this task[17, 18]. System controllability under Kraus-map, which includes non-projective measurements, was also studied[19, 20]. More extreme cases with measurement-only control schemes have been studied. In the absence of active coherent controls, projective measurements were introduced to map an unknown mixed state onto a known expected pure state while only two noncommuting observables are available[21]. Optimized measurement sequence was obtained for two-level quantum systems[22, 23]. The continuous measurement extension was investigated and it shows that quantum anti-Zeno effect could be recovered in the limit of continuous

cases[4, 23–26]. Schemes for state engineering using measurements and fixed dynamics were also proposed[27]. Recently, a novel quantum state preparing protocol was presented in which only a restricted set of measurements is required[28]. In this protocol, previous measurement outcome is feedback as the control, and it shows that arbitrary state could indeed be prepared. This feedback idea was also introduced to state manipulation[29].

These above works investigated both two-level[15, 20, 22, 23, 28] and N -level systems[5, 14, 17, 19, 21, 27, 30]. In this paper, we will investigate how to control N -level systems with arbitrary projective measurements only, which is a direct extension of our previous works[30]. However, powerful tools in two-level context, i.e. Pauli matrices and Stokes vector, are no longer valid in N -level systems with $N > 2$. In Ref[30], a variational method over unitary group was introduced to obtain the necessary condition for optimal measurement sequence. This condition was carefully analyzed for pure initial and expected states with a single measurement. This will be extended to arbitrary finite times of projective measurements for pure state-to-state control. We find that the control space will be reduced to an effective subspace spanned by the initial and expected states only. The transition cases through intermediate levels are excluded and the two-level model[22, 23] will be recovered.

The rest of the paper is organized as follows. In Sec. II, we introduce some basic notations, the variational method over unitary group and the necessary condition for an optimal measurement sequence under non-selective assumption. In Sec. III, we investigate an orthogonal state-to-state control case, and derive the effective subspace for optimal control. These cases will be generalized to ones of non-orthogonal pure states in Sec. IV. Discussion and conclusion are presented in Sec.V.

II. NECESSARY CONDITION FOR OPTIMAL STATE TRANSFER

The measurement-based optimal control case for a two-level quantum system was solved by using Pauli matrices, and the optimal projectors were also obtained[22]. Since this approach seems invalid for general N -level systems, we introduce a virational method and derive a necessary condition for the optimal measurement sequence[30]. In this section, some basic concepts and notations are firstly be introduced. Then, the virational method and the necessary condition for the optimal measurement sequence, which appears as a chain of equalities, are presented. These tools will be utilized to study N -level cases in the next section.

Considering an N -level quantum system whose states is characterized by a density matrix ρ , the state after a non-selective measurement \mathcal{M} becomes $\mathcal{M}(\rho) = \sum_{i=1}^N P_i \rho P_i$ where $\{P_i\}$ is a set of rank-1 projectors satisfying the equations $P_i P_j = \delta_{ij} P_i$ and $\sum_{i=1}^N P_i = I$. Because of these properties, we could always find a suitable unitary matrix U to diagonalize these projectors, i.e. $P_i = U |i\rangle \langle i| U^\dagger$ for $i = 1, \dots, N$, where $|i\rangle$ indicates a specific level of the system. With these notations, the non-selective measurement process could be recast as

$$\mathcal{M}(U) \rho = \sum_{i=1}^N U |i\rangle \langle i| U^\dagger \rho U |i\rangle \langle i| U^\dagger \quad (1)$$

which contains three successive operations of ρ : (1) rotate the basis (a unitary transformation) (2) set non-diagonal elements to be zero (3) rotate the basis inversely.

Limited by the measurement techniques nowadays, it is difficult to perform too many projective measurements within finite time interval. Assuming that at most m -times measurement could be employed in this interval, the final state ρ^m after these measurements reads $\rho^m = \mathcal{M}(U_m) \dots \mathcal{M}(U_1) \rho$. We neglect free evolution of the system because this could be included via a picture transformation[22, 23]. The control objective J , which quantifies this control effect, is defined as the overlap between ρ^m and some expected final state θ :

$$J = \text{Tr}(\rho^m \theta) = \text{Tr}((\mathcal{M}(U_m) \dots \mathcal{M}(U_1) \rho) \theta). \quad (2)$$

The necessary condition for optimal control sequence is obtained by considering the first derivative of J with respect to variables $\{U_i\}$. A variational analysis method is introduced to parametrize the neighborhood of a unitary transformation and the necessary condition presents as a chain of equalities:

$$\begin{aligned} & [\rho, \mathcal{M}(U_1) \dots \mathcal{M}(U_m) \theta] \\ &= \dots \\ &= [\mathcal{M}(U_k) \dots \mathcal{M}(U_1) \rho, \mathcal{M}(U_{k+1}) \dots \mathcal{M}(U_m) \theta] \\ &= \dots \\ &= [\mathcal{M}(U_m) \dots \mathcal{M}(U_1) \rho, \theta]. \end{aligned} \quad (3)$$

Detailed mathematical derivation of the above equalities and some special cases for single measurement was shown in [30], but general cases with m -times measurements are still intractable. The rest of this paper will focus on this general scenario.

III. EFFECTIVE SUBSPACE FOR OPTIMAL ORTHOGONAL PURE STATES TRANSFER

In this section, the effective subspace for orthogonal pure state-to-state optimal measurement control will be investigated. Assuming that the initial and expected states are $\rho = |1\rangle \langle 1|$ and $\theta = |2\rangle \langle 2|$, respectively, and all states are represented in the basis $\{|i\rangle, i = 1, 2, \dots, N\}$. Due to the special form of ρ , $[\rho, \mathcal{M}(U_1) \dots \mathcal{M}(U_m) \theta]$ becomes a matrix with non-zero elements only in the first row and first column. By the same reason, $[\mathcal{M}(U_m) \dots \mathcal{M}(U_1) \rho, \theta]$ becomes a matrix with non-zero elements only in the second row and second column. Combining these two forms, we have

$$\begin{aligned} & [\rho, \mathcal{M}(U_1) \dots \mathcal{M}(U_m) \theta] \\ &= \dots \\ &= [\mathcal{M}(U_m) \dots \mathcal{M}(U_1) \rho, \theta] \\ &= \begin{bmatrix} 0 & c & O \\ -c^* & 0 & O \\ O & O & O \end{bmatrix} = C \end{aligned} \quad (4)$$

where c is a complex number and O is a zero matrix of suitable size. The case of $c = 0$ is dropped henceforth, since it corresponds to a trivial control which can not be the optimal one.

To clarify the discussion below, some notations are introduced here: I^n denotes $n \times n$ identity matrix; $U(n)$ denotes n -dimensional unitary group and U^n represents some elements in $U(n)$; O^n is $n \times n$ zero matrix; $D(n)$ is the set of $n \times n$ density matrices, i.e. positive semidefinite Hermitian matrices. With these notations, we will first investigate the symmetry property of $\{U_i\}$ and propose lemma.1.

Lemma.1 if $\{U_1, U_2, \dots, U_m\}$ is an optimal control sequence, then $\{U_s U_1, U_s U_2, \dots, U_s U_m\}$ could also achieve this optimum, where $U_s = e^{i\psi_1} \oplus e^{i\psi_2} \oplus U_{arb}$. Here, ψ_1, ψ_2 are arbitrary phases and $U_{arb} \in U(N-2)$.

The proof of lemma.1 is straightforward. Assuming that $\{U_s^\dagger U_1, U_s^\dagger U_2, \dots, U_s^\dagger U_m\}$ could also achieve this optimum, then we have sufficient conditions for optimal control: (i) $U_s^{l\dagger} U_s^{l+1} = I$ for $l = 1, \dots, m-1$ and (ii) ρ and θ are invariant under the action of U_s^1 and U_s^m , respectively. Consequently, we find that $U_s^1 = \dots = U_s^m$ and $U_s^1 = U_s^m = e^{i\psi_1} \oplus e^{i\psi_2} \oplus U_{arb}$.

According to lemma.1, we could always find proper ψ_1 or ψ_2 to eliminate the phase in c and make it real. Thus, c will be treated as a real number hereafter.

Now, we investigate the first two parts of Eq.(4) and

get

$$[\rho, \mathcal{M}(U_1)\tau] = C \quad (5)$$

$$[\mathcal{M}(U_1)\rho, \tau] = C \quad (6)$$

in which τ is defined as $\mathcal{M}(U_2)\dots\mathcal{M}(U_m)\theta$. Multipling U_1^\dagger and U_1 from left and right in Eqs.(5) and (6), and taking the (p, q) matrix element of both sides, we have

$$\begin{aligned} & (U_1)_{1p}^* (U_1)_{1q} \left((U_1^\dagger \tau U_1)_{qq} - (U_1^\dagger \tau U_1)_{pp} \right) \\ &= ((U_1)_{1p}^* (U_1)_{2q} - (U_1)_{2p}^* (U_1)_{1q}) c \end{aligned} \quad (7)$$

$$\begin{aligned} & (U_1^\dagger \tau U_1)_{pq} ((U_1)_{1p}^* (U_1)_{1q} - (U_1)_{1q}^* (U_1)_{1p}) \\ &= ((U_1)_{1p}^* (U_1)_{2q} - (U_1)_{2p}^* (U_1)_{1q}) c \end{aligned} \quad (8)$$

Obviously, Eq.(7) implies that if $(U_1)_{1p} \neq 0$ and $(U_1)_{1q} \neq 0$ then $(U_1^\dagger \tau U_1)_{qq} - (U_1^\dagger \tau U_1)_{pp}$ could be expressed by $\{(U_1)_{1i}\}$ and $\{(U_1)_{2i}\}$ explicitly. Further, we can write the objective J as $Tr(\mathcal{M}(U_1)\rho\tau) = \sum_i (U_1)_{1i} (U_1)_{1i}^* (U_1^\dagger \tau U_1)_{ii}$. Note that $\{(U_1^\dagger \tau U_1)_{ii}\}$ could be reconstructed by $\{(U_1)_{1i}\}$ and $\{(U_1)_{2i}\}$ up to a constant, and if we fix this constant, which corresponds to solving Eq.(7) and Eq.(8) in a special case, then $(U_1^\dagger \tau U_1)_{ii}$ will only depend on $\{(U_1)_{1i}\}$ and $\{(U_1)_{2i}\}$, and so does J . This observation will give us the explicit form of τ .

Using lemma.1, we could diagonalize the submatrix spanned by the last $N-2$ rows and columns in τ , and then investigate $(U_1^\dagger \tau U_1)_{11}$. Since $(U_1^\dagger \tau U_1)_{11}$ only depends on $\{(U_1)_{1i}\}$ and $\{(U_1)_{2i}\}$, we have $\frac{\partial (U_1^\dagger \tau U_1)_{11}}{\partial (U_1)_{31}} = \frac{\partial \sum_{ij} (U_1)_{1i}^* (U_1)_{j1} \tau_{ij}}{\partial (U_1)_{31}} = 0$. Again, by lemma.1, we can assign some U_{arb} to make $\{(U_1)_{1i}\}$ real for $i = 3, \dots, N$. This will simplify our calculation below. Remember that there is a normalization constraint, i.e. $\sum_i (U_1)_{i1} (U_1)_{i1}^* = 1$, which implies that $\{(U_1)_{i1}\}$ are not independent in $(U_1^\dagger \tau U_1)_{11}$. Due this constraint, we treat $(U_1)_{N1}$ as a non-independent variable and have

$$\begin{aligned} & \frac{\partial \sum_{ij} (U_1)_{1i}^* (U_1)_{j1} \tau_{ij}}{\partial (U_1)_{31}} \\ &= \frac{\partial ((U_1)_{11} \tau_{31} + (U_1)_{21} \tau_{32} + c.c.) (U_1)_{31}}{\partial (U_1)_{31}} \\ &+ \frac{\partial ((U_1)_{11} \tau_{N1} + (U_1)_{21} \tau_{N2} + c.c.) (U_1)_{N1}}{\partial (U_1)_{31}} \\ &+ \frac{\partial ((U_1)_{31}^* (U_1)_{31} \tau_{33} + (U_1)_{N1}^* (U_1)_{N1} \tau_{NN})}{\partial (U_1)_{31}} \\ &= 0 \end{aligned} \quad (9)$$

Since $(U_1)_{11}, (U_1)_{11}^*, (U_1)_{21}, (U_1)_{21}^*$ are independent, three parts in Eq.(9) gives $\tau_{13} = \tau_{31} = \tau_{23} = \tau_{32} = 0$, $\tau_{N1} = \tau_{N2} = \tau_{1N} = \tau_{2N} = 0$ and $\tau_{33} = \tau_{NN}$. By the same reason, we also have $\tau_{1i} = \tau_{2i} = 0$ and $\tau_{ii} = \tau_{NN}$ if we investigate $\frac{\partial (U_1^\dagger \tau U_1)_{11}}{\partial (U_1)_{i1}}$ for $i = 3, \dots, N$. This implies $\tau \in D(2) \oplus dI^{N-2}$ where d is a normalized factor.

Next, we start from this special form of τ to explore what kind of $\{U_i\}$ could achieve the optimum. We define $\tau = \tau_s \oplus dI^{N-2}$ where $\tau_s \in D(2)$ and write $\mathcal{M}(U_1)\rho$ in a block form, i.e. $\mathcal{M}(U_1)\rho = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, in which A_{11} and A_{22} are 2×2 and $(N-2) \times (N-2)$ matrices respectively. According to Eq.(6), we get $(dI^2 - \tau_s)A_{12} = O$ whose solutions have two different branches: (a) τ_s has no eigenvalue of d or (b) τ_s has an eigenvalue of d , and these two branches indicate A_{12} is zero or not, respectively.

For case (a), we have $A_{12} = O$ and $\mathcal{M}(U_1)\rho = \begin{bmatrix} A_{11} & O \\ O & A_{22} \end{bmatrix}$. To acquire this form of $\mathcal{M}(U_1)\rho$, U_1 must be (a.1) $U_1 \in (U(2) \oplus U(N-2)) \times (1 \oplus U(N-1))$ or (a.2) $U_1 \in U(2) \oplus U(N-2)$. Note that these two different cases origin from whether A_{11} and A_{22} have degenerated eigenvalue.

For (a.1), we define $U_1 = V_1 W_1$ in which $V_1 \in U(2) \oplus U(N-2)$ and $W_1 \in 1 \oplus U(N-1)$. Since the rotation V_1 makes ρ and τ belong to $D(2) \oplus O^{N-2}$ and $D(2) \oplus dI^{N-2}$ respectively, a further rotation W_1 makes some diagonal elements, except the first one, in $V_1^\dagger \rho V_1$ equal, which are the degenerated eigenvalue in A_{11} and A_{22} . W_1 will also make the diagonal elements in the corresponding subspace of $V_1^\dagger \tau V_1$ equal. This is due to the fact that this two subspaces, i.e. $diag(a, 0, \dots, 0)$ and $diag(b, d, \dots, d)$ are essentially of the same form of $\alpha diag(a, 0, \dots, 0) + \beta diag(1, 1, \dots, 1)$ which will have same diagonal elements under the act of W_1 . This property also implies that W_1 must act on the whole $(N-1)$ -dimensional subspace to achieve the maximum. Define the rotation with this property by $U_e(N)$, we get $U_1 \in (U(2) \oplus U(N-2)) \times (1 \oplus U_e(N-1))$. Now, we have $U_2 \in U(2) \oplus U(N-2)$ from (a) and $U_1 \in (U(2) \oplus U(N-2)) \times (1 \oplus U_e(N-1))$. This implies $\mathcal{M}(U_2)\mathcal{M}(U_1)\rho \in D(2) \oplus d'I^{N-2}$ where no degenerated eigenvalue exists between these two subspaces. According to Eq. (4), it indicates that $\mathcal{M}(U_3)\dots\mathcal{M}(U_m)\theta \in D(2) \oplus dI^{N-2}$. Again, we have U_3 belongs to $U(2) \oplus U(N-2)$ or $(U(2) \oplus U(N-2)) \times (1 \oplus U_e(N-1))$. The latter one, i.e. $U_3 \in (U(2) \oplus U(N-2)) \times (1 \oplus U_e(N-1))$, implies that two eigenvalues in $\mathcal{M}(U_2)\dots\mathcal{M}(U_m)\theta$, which are not d , are both larger or smaller than d . If they are both larger than d , then U_1 will belong to $U(2) \oplus U(N-2)$ but not $(U(2) \oplus U(N-2)) \times (1 \oplus U_e(N-1))$. If they are both smaller than d , then the objective could not exceed d which is less than $1/2$, and this is contrary to the known optimum[22, 23]. Thus, we have $U_3 \in U(2) \oplus U(N-2)$. This procedure could be operated in an iterative fashion, and we finally have $U_i \in U(2) \oplus U(N-2)$ for $i = 2, \dots, N$. Note that if we choose $U_1 = V_1$ and discard W_1 , then J will be larger than the one when we have $U_1 = V_1 \times W_1$. Thus we have $U_i \in U(2) \oplus U(N-2)$ for $i = 1, \dots, N$.

If we have (a.2), then both U_1 and U_2 belong to $U(2) \oplus U(N-2)$, and Eq.(4) implies that $\mathcal{M}(U_3)\dots\mathcal{M}(U_m)\theta \in D(2) \oplus D(N-2)$. By considering the special form of τ , we also have $\mathcal{M}(U_3)\dots\mathcal{M}(U_m)\theta \in D(2) \oplus dI^{N-2}$

which indicates that $U_3 \in U(2) \oplus U(N-2)$ or $U_3 \in (U(2) \oplus U(N-2)) \times (1 \oplus U(N-1))$. For the former one, we could consequently investigate U_4 which revives the discussion of (a.2). And the later one is just case(b) when we treat U_3 in (a.2) as U_2 in case (b). We will also show that U_1 in case(b) also belongs to $U(2) \oplus U(N-2)$ as U_2 in (a.2). So, this situation, i.e. (a.2), is equivalent to case(b).

For case (b), the optimal control between τ and ρ is equivalent to the optimal control between two non-orthogonal pure states. This could be seen when we diagonalize τ and get $\text{diag}[a, d, \dots, d]$. The solution of this case is obtained[30] and we have $U_1 \in U(2) \oplus U(N-2)$. Define $U_2 = V_2 W_2$ where $V_2 \in U(2) \oplus U(N-2)$, $W_2 \in 1 \oplus U_e(N-1)$, and

$$\begin{aligned} \chi &= V_2^\dagger \mathcal{M}(U_3) \dots \mathcal{M}(U_m) \theta V_2 \\ &= \begin{bmatrix} B_{11} & B_{21,up} \\ B_{21,up}^\dagger & B_{21,down}^\dagger \\ B_{21,up}^\dagger & B_{21,down}^\dagger \\ B_{22} \end{bmatrix} \end{aligned}$$

where $B_{21,up}$ and $B_{21,down}$ are $(N-2)$ dimensional row vectors. Since arbitrary $U \in I^2 \oplus U(N-2)$ will not change the diagonal elements in $V_2^\dagger \mathcal{M}(U_1) \rho V_2$ and a successive W_2 makes diagonal elements, except the first one, in $V_2^\dagger \mathcal{M}(U_1) \rho V_2$ equal, we find that $B_{22} = dI^{N-2}$ and $B_{21,down} = 0$. Define $\kappa = \sum_{i=1}^N W_2 |i\rangle \langle i| W_2^\dagger V_2^\dagger \mathcal{M}(U_1) \rho V_2 W_2 |i\rangle \langle i| W_2^\dagger$, and Eq.(4) gives $\kappa \chi - \chi \kappa = V_2^\dagger C V_2$ which implies $B_{21,up} = 0$. Thus we have $\mathcal{M}(U_3) \dots \mathcal{M}(U_m) \theta \in D(2) \oplus d'' I^{N-2}$ and again, $U_3 \in U(2) \oplus U(N-2)$ or $U_3 \in (U(2) \oplus U(N-2)) \times (1 \oplus U_e(N-1))$. The latter one should be discarded since it will not achieve the maximum when $U_2 \in (U(2) \oplus U(N-2)) \times (1 \oplus U(N-1))$, and finally we have $U_3 \in U(2) \oplus U(N-2)$. Following the analysis of (a.1) above, we also have $U_i \in U(2) \oplus U(N-2)$ for $i = 1, \dots, N$.

To conclude, for orthogonal pure states control induced by projective measurements only, the control space will be reduced to a two-dimensional effective subspace which spanned by the initial and expected states only.

IV. GENERALIZATION TO NON-ORTHOGONAL PURE STATES

The analysis in Sec.III could be generalized to cases with non-orthogonal pure states, and Eq.(4) is still valid. Define these two pure states by $\rho = |1\rangle \langle 1|$ and $\theta = (\alpha |1\rangle + \beta |2\rangle) (\alpha^* \langle 1| + \beta^* \langle 2|)$. Since $\rho = |1\rangle \langle 1|$, we have

$$[\rho, \mathcal{M}(U_1) \dots \mathcal{M}(U_m) \theta] = \begin{bmatrix} 0 & c & s \\ c^* & 0 & O \\ s^\dagger & O & O \end{bmatrix} \text{ where } c \text{ is a com-}$$

plex number and s is a $(N-2)$ -dimensional row vector. On the other hand, if we assume that θ could be diagonalized by u , i.e. $u \theta u^\dagger = |1\rangle \langle 1|$, then

$$\begin{aligned} &(u \oplus I^{N-2}) \begin{bmatrix} 0 & c & s \\ c^* & 0 & O \\ s^\dagger & O & O \end{bmatrix} (u^\dagger \oplus I^{N-2}) \\ &= (u \oplus I^{N-2}) [\mathcal{M}(U_m) \dots \mathcal{M}(U_1) \rho, \theta] (u^\dagger \oplus I^{N-2}) \\ &= [\mathcal{M}((u \oplus I^{N-2}) U_m) \dots \mathcal{M}((u \oplus I^{N-2}) U_1) \\ &\quad ((u \oplus I^{N-2}) \rho (u^\dagger \oplus I^{N-2})), |1\rangle \langle 1|] \end{aligned} \quad (10)$$

The right hand of Eq.(10) indicates that it will be a matrix with non-zero elements in the first row and col-

umn only, thus $(u \oplus I^{N-2}) \begin{bmatrix} 0 & c & s \\ c^* & 0 & O \\ s^\dagger & O & O \end{bmatrix} (u^\dagger \oplus I^{N-2}) =$

$\begin{bmatrix} 0 & c' & s' \\ c'^* & 0 & O \\ s'^\dagger & O & O \end{bmatrix}$. Since u is nontrivial, we must have $s = O$

and Eq.(4) is recovered. The rest of the proof for non-orthogonal pure states is the same as that in Sec.III. We could also conclude that the effective subspace will be reduced to the one spanned by $|1\rangle$ and $|2\rangle$.

V. DISCUSSION AND CONCLUSION

In summary, we discuss the use of projective measurements to control an N -level quantum system. Previous works of two-level scenario are generalized assuming that the measurement time is limited. It is proved that for pure initial and expected states, the control space will be reduced to an effective subspace spanned by these two states. The intuition for this result is straightforward, no extra level is needed for optimal control. The situation for mixed states is much complicated. Numerical calculations show that this conclusion still works if “pure states” are replaced by “mixed states”, and its rigorous proof will be further investigated.

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